CS103
Winter 2025

Lecture 06:

Functions

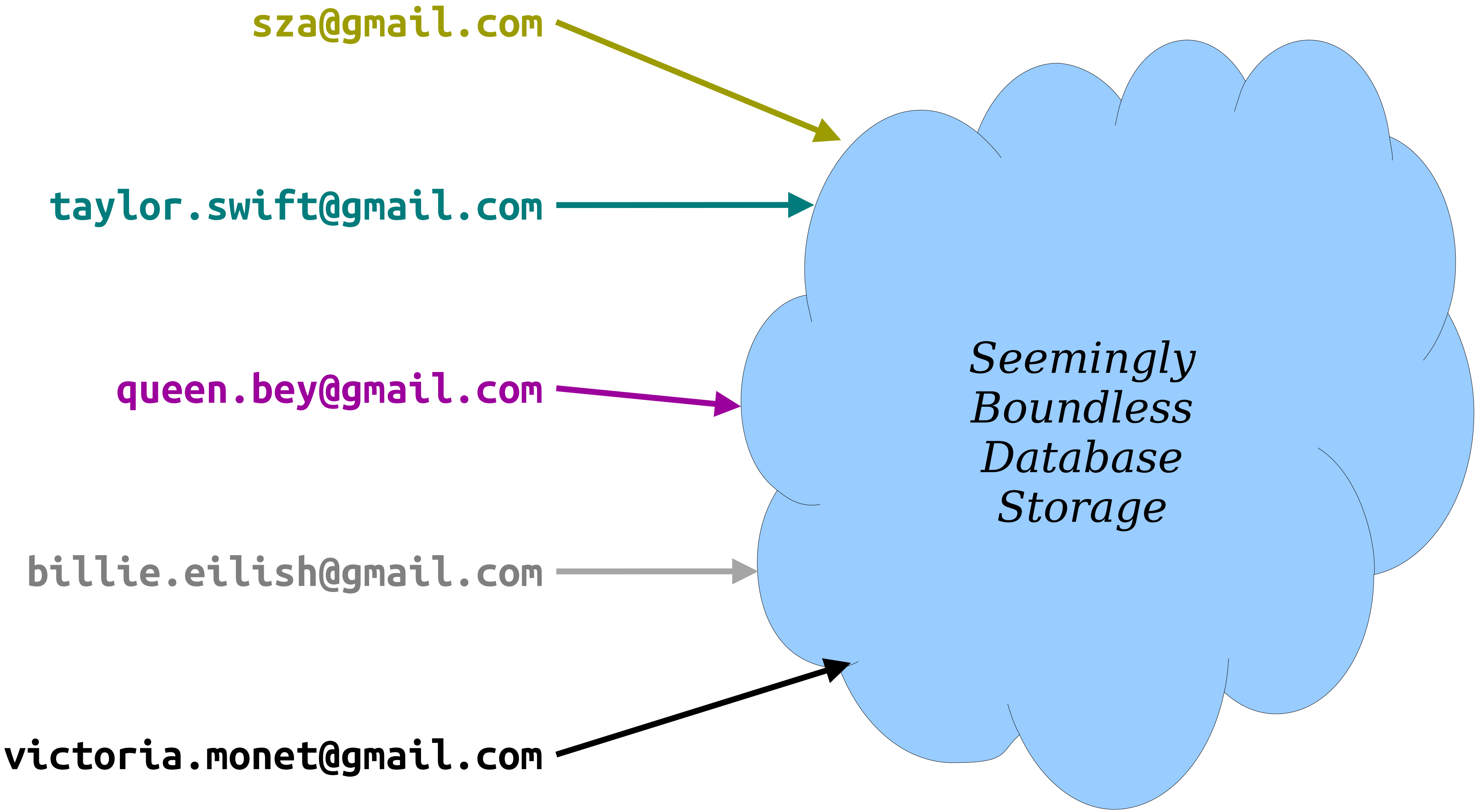
Part 1 of 2

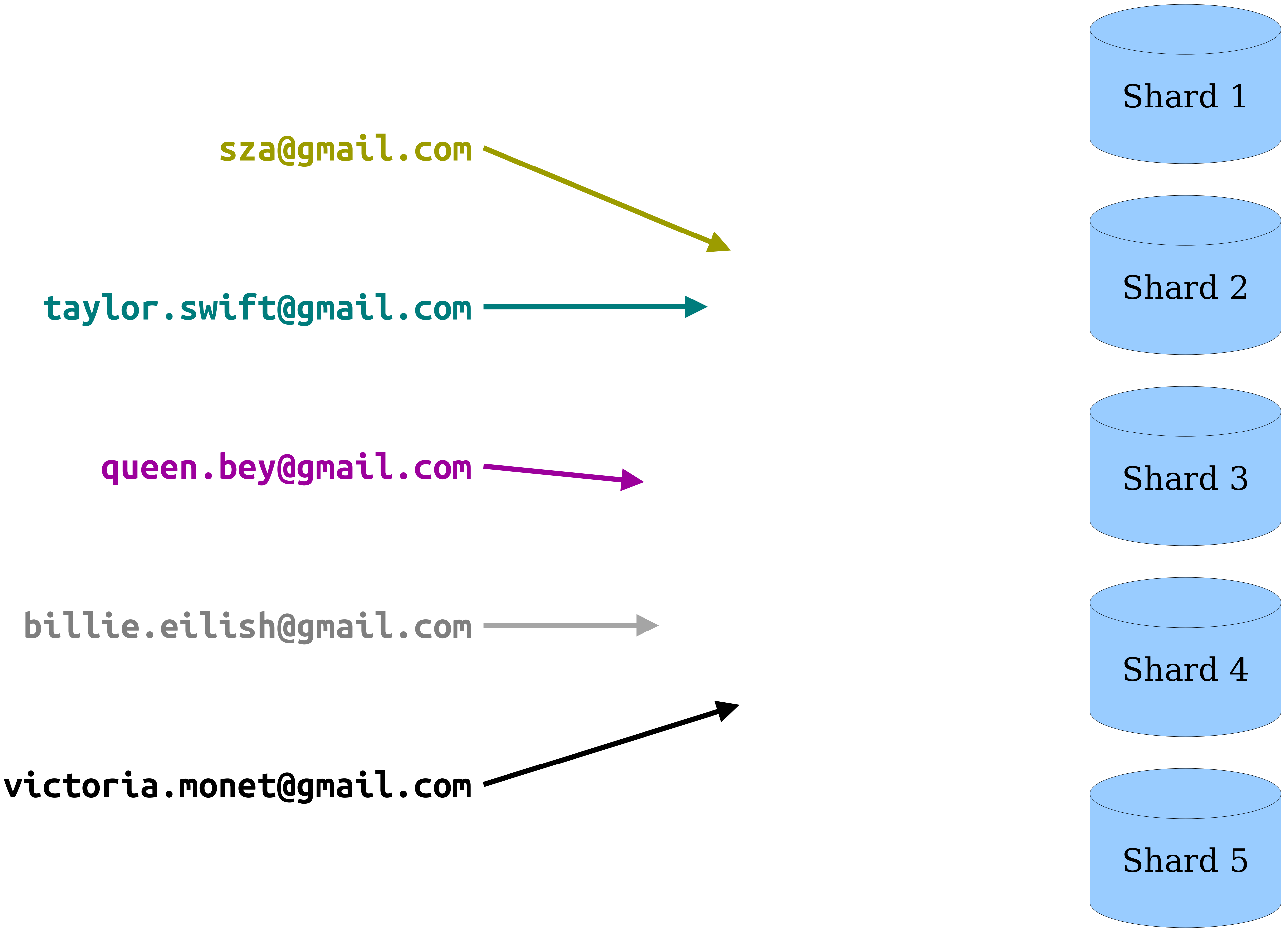
Outline for Today

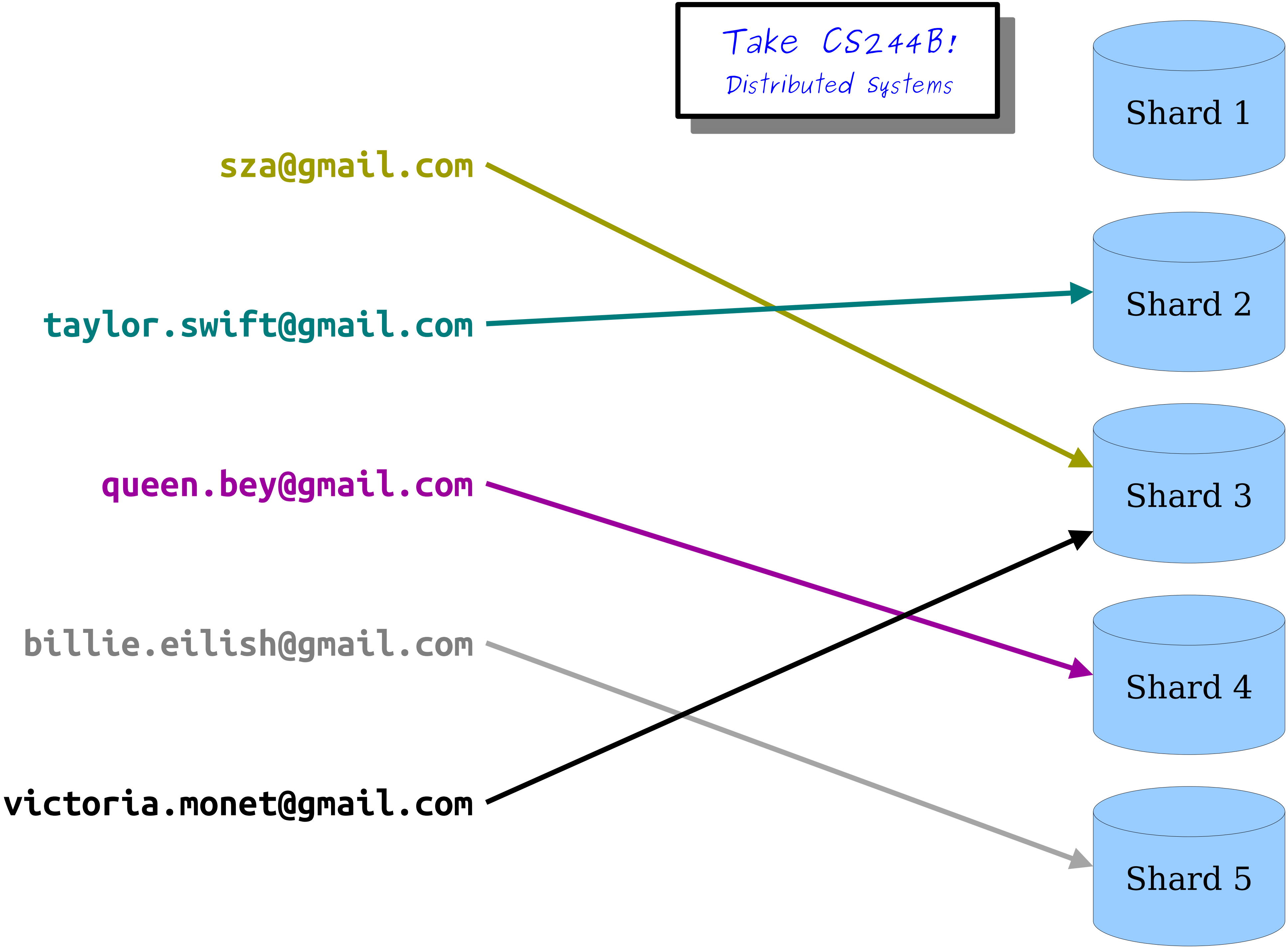
- ***What is a Function?***
  - It's more nuanced than you might expect.
- ***Domains and Codomains***
  - Where functions start, and where functions end.
- ***Defining a Function***
  - Expressing transformations compactly.
- ***Special Classes of Functions***
  - Useful types of functions you'll encounter IRL.
- ***Proofs on First-Order Definitions***
  - A key skill.

What is a function?

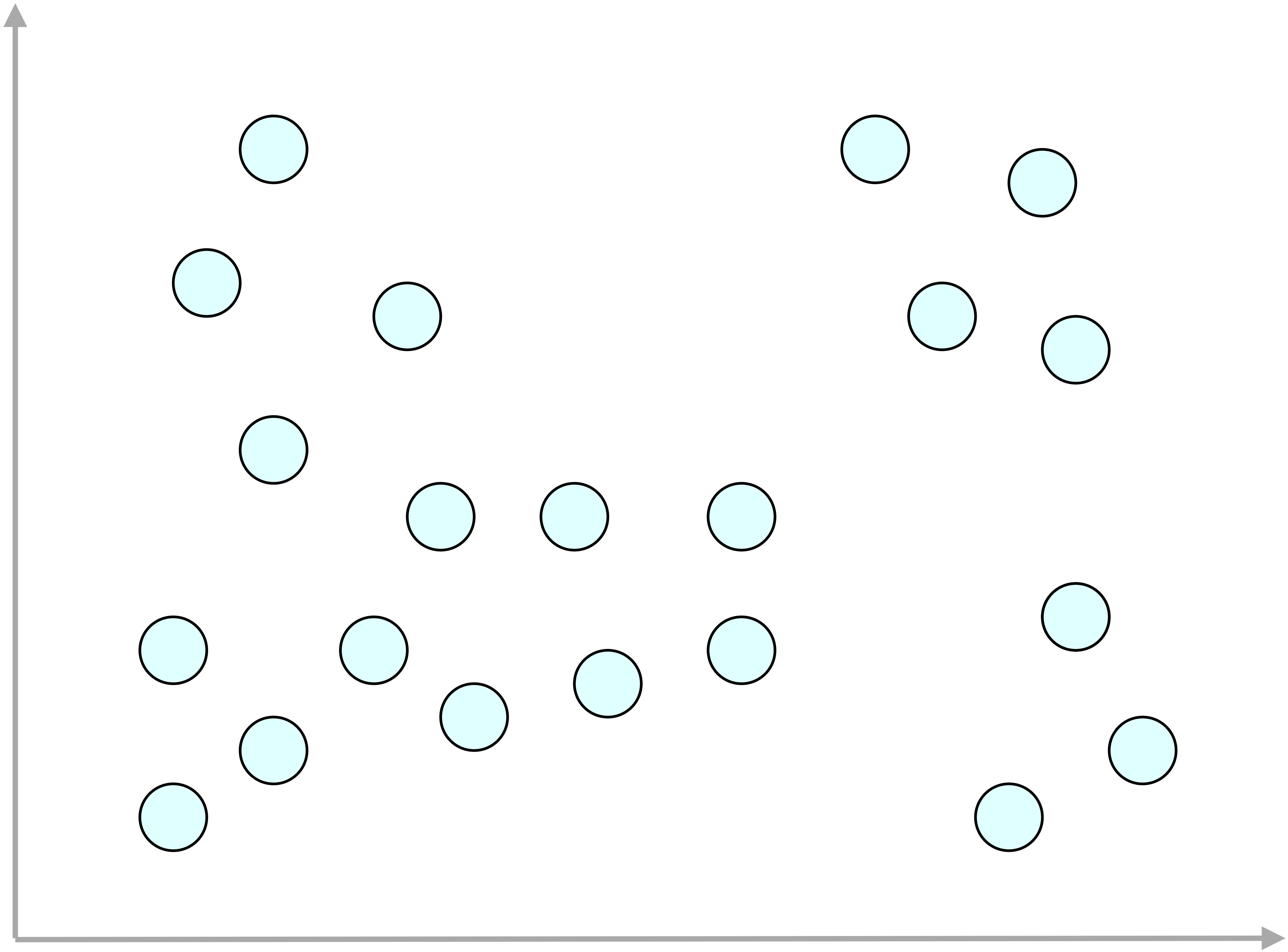
Motivating Example 1: Database Sharding

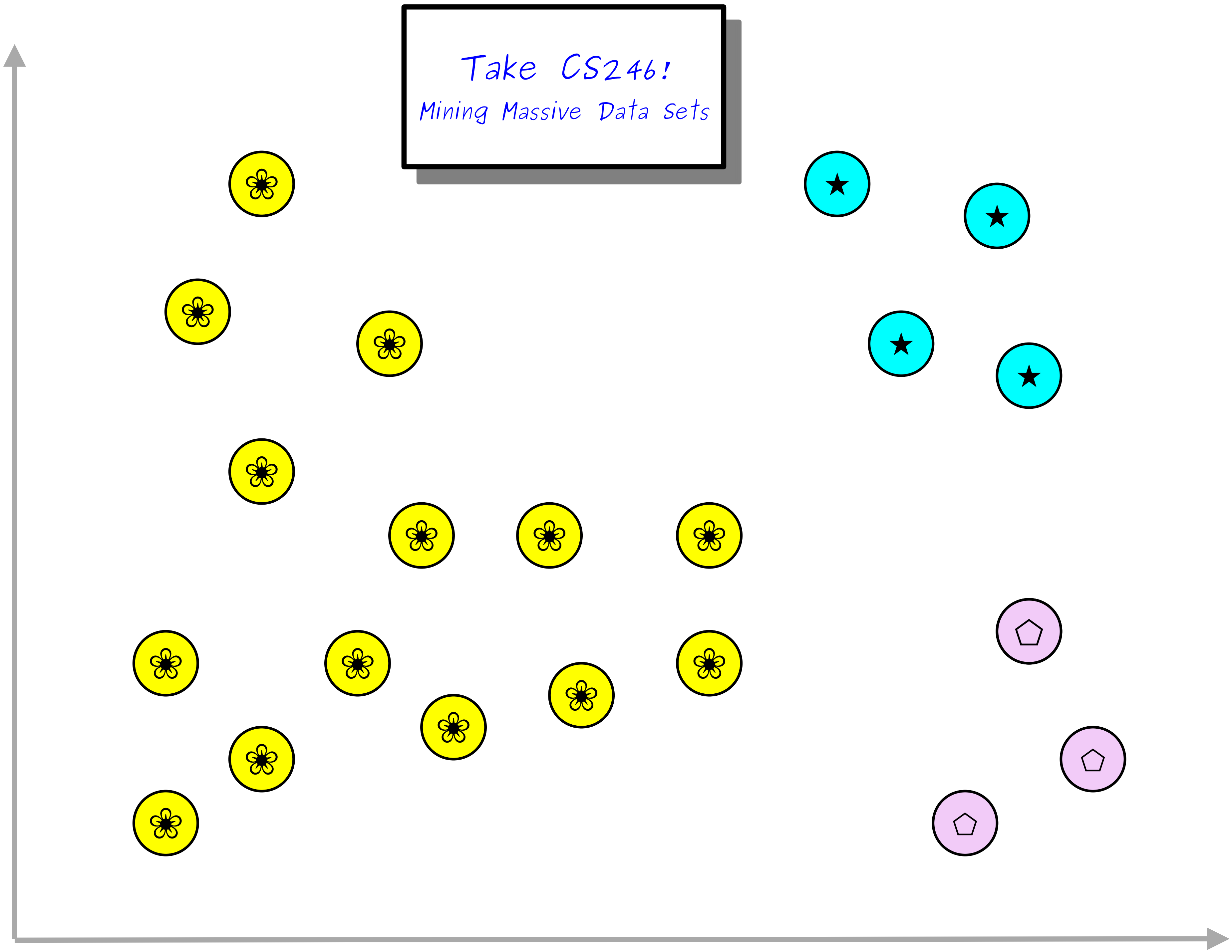

sza@gmail.com
taylor.swift@gmail.com
queen.bey@gmail.com
billie.eilish@gmail.com
victoria.monet@gmail.com
Seemingly Boundless Database Storage


sza@gmail.com
taylor.swift@gmail.com
queen.bey@gmail.com
billie.eilish@gmail.com
victoria.monet@gmail.com
Shard 1
Shard 2
Shard 3
Shard 4
Shard 5


Take CS244B!
Distributed Systems
sza@gmail.com
taylor.swift@gmail.com
queen.bey@gmail.com
billie.eilish@gmail.com
victoria.monet@gmail.com
Shard 1
Shard 2
Shard 3
Shard 4
Shard 5

Motivating Example 2: Data Clustering



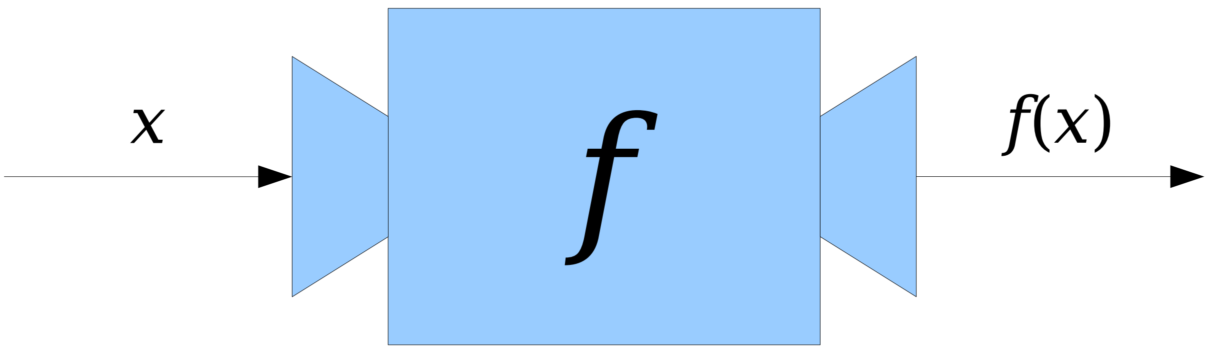

Take CS246!
Mining Massive Data Sets

What's In Common?

- We have a fixed, known set of possible inputs.
  - In our examples: user names and 2D data points
- We have a fixed, known set of possible outputs.
  - In our examples: database shards and cluster labels.
- Each input is assigned an output.
  - Some outputs might be assigned multiple inputs.
  - Some outputs might be assigned no inputs.

High-Level Intuition:

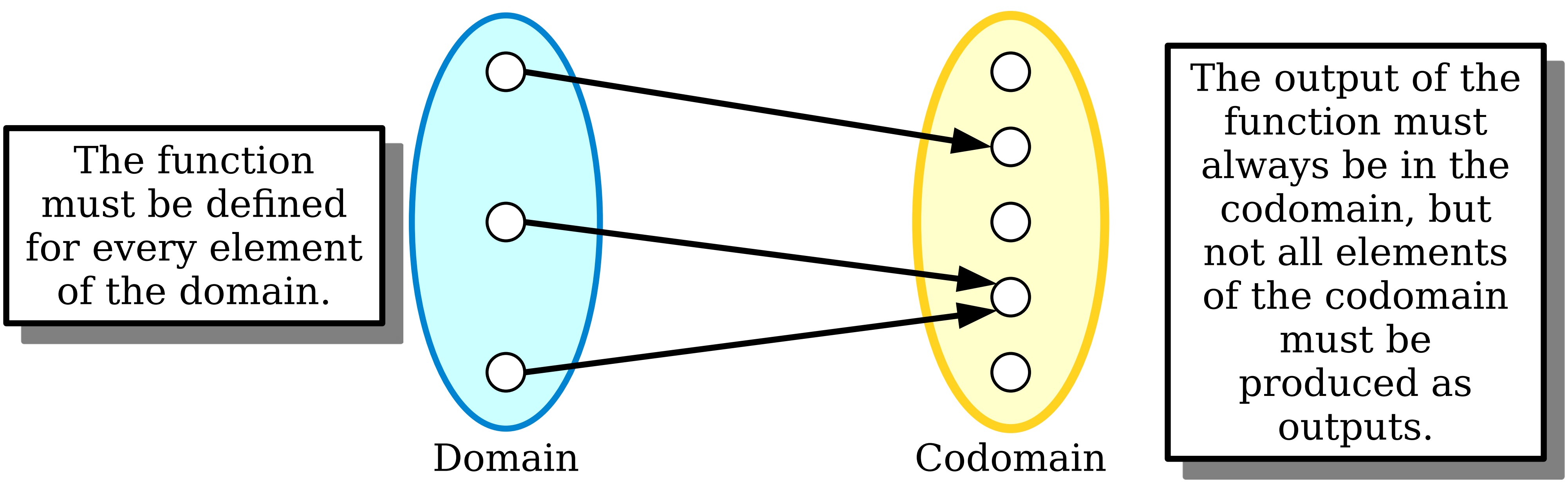
A function is an object f that takes in exactly one input x and produces exactly one output $f(x)$.



(This is not definition. It's just to help you build and intuition.)

Domains and Codomains

- Every function *f* has two sets associated with it: its ***domain*** and its ***codomain***.
- A function *f* can only be applied to elements of its domain. For any *x* in the domain, *f*(*x*) belongs to the codomain.



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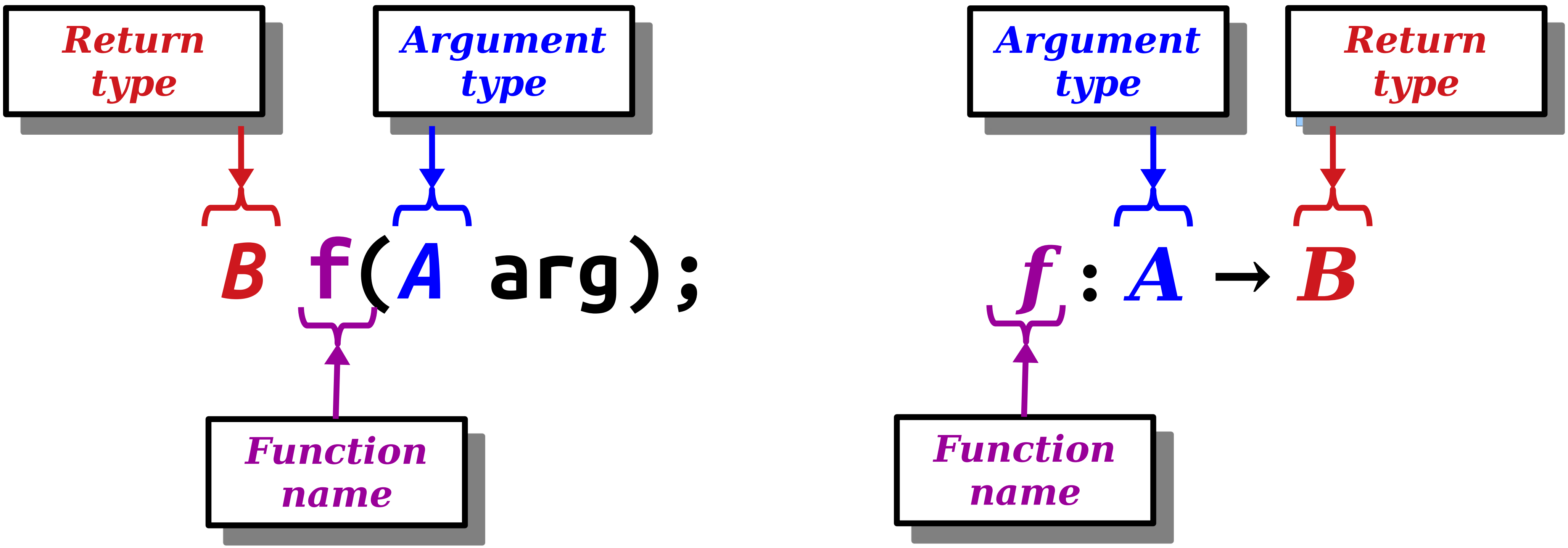
The ***domain*** of this function is ℝ. Any real number can be provided as input.

The ***codomain*** of this function is ℝ. Everything produced is a real number, but not all real numbers can be produced.

```
double absoluteValueOf(double x) {
    if (x >= 0) {
        return x;
    } else {
        return -x;
    }
}
```

Domains and Codomains

- If f is a function whose domain is A and whose codomain is B, we write $\boldsymbol{f : A \to B}$.
- Think of this like a “function prototype” in C++.



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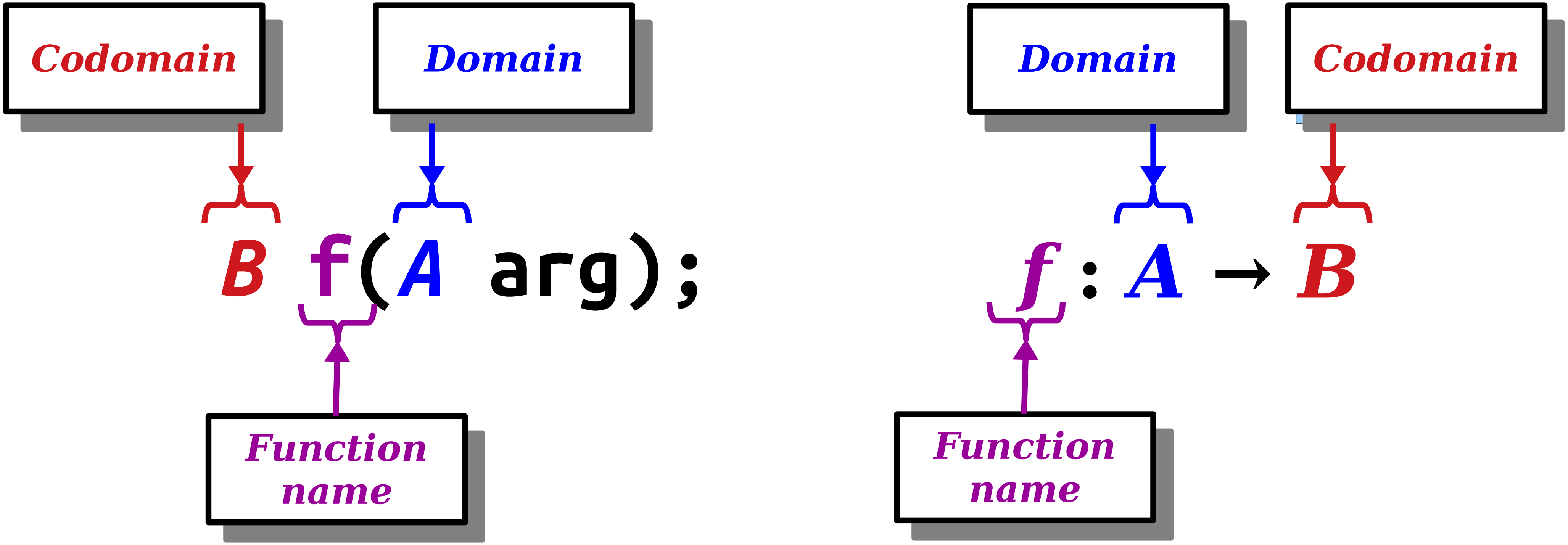


Some Observations

- Usually, when working with functions, you pick the domain and codomain before defining the rule for the function.
  - Think programming: you usually know what types of things you're working with before you know how they work.
- In mathematics, all functions take in exactly one argument: an element of the domain.
  - If you're clever, you can get two or more arguments to a function while still obeying this rule. Chat with me after class to learn more!
- In mathematics, functions are ***deterministic*** and can't behave randomly.
  - If you're clever, you can get functions that kinda sorta ish look random. Chat with me after class to learn more!

The Official Rules for Functions

- Formally speaking, we say that $f : A \to B$ if the following two rules hold.
- First, f must be obey its domain/codomain rules:

$$\forall a \in A.\ \exists b \in B.\ f(a) = b$$

(*"Every input in A maps to some output in B."*)

- Second, f must be deterministic:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 = a_2 \to f(a_1) = f(a_2))$$

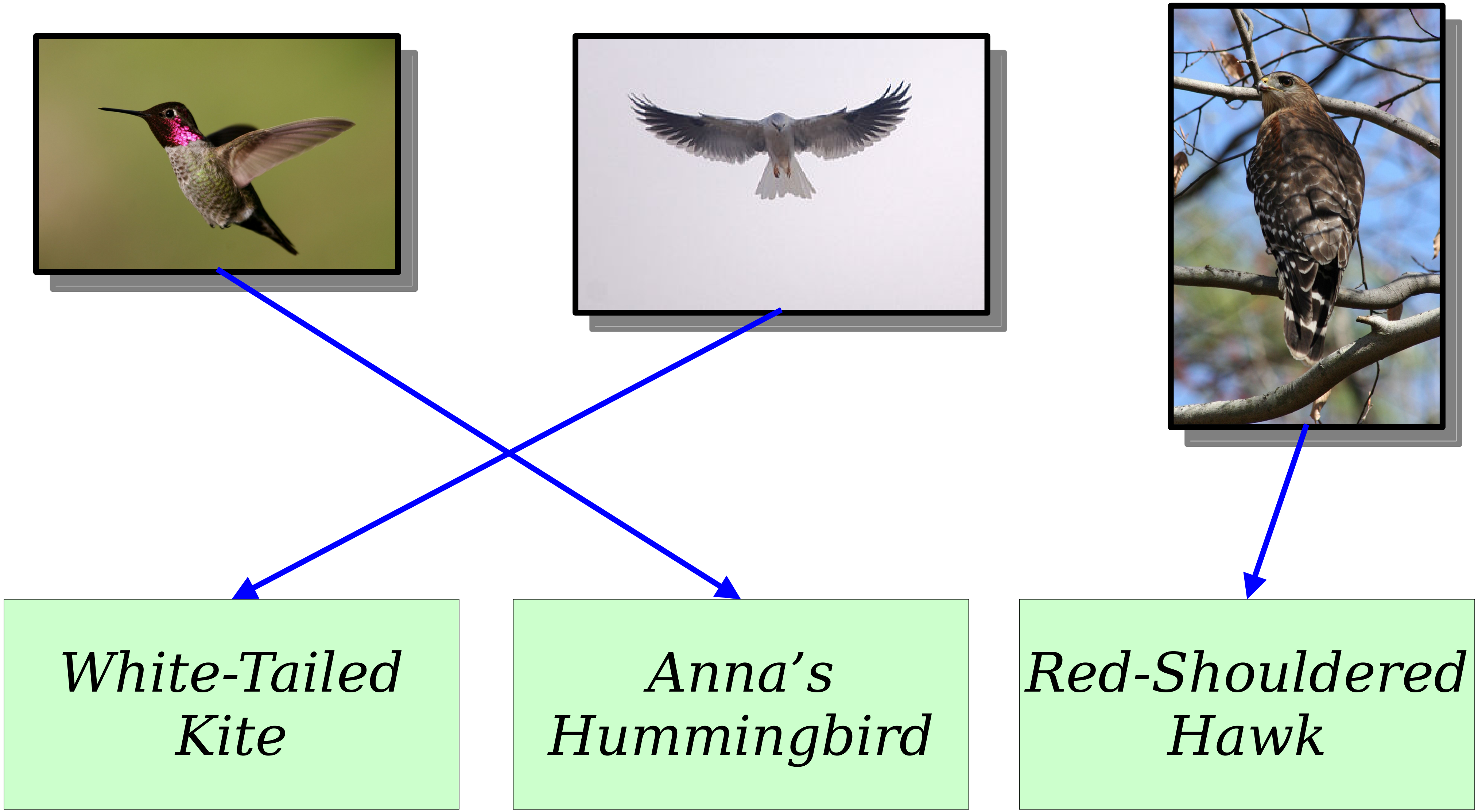
(*"Equal inputs produce equal outputs."*)

- If you're ever curious about whether something is a function, look back at these rules and check! For example:
  - Can a function have an empty domain?
  - Can a function have an empty codomain?

Defining Functions

Defining Functions

- To define a function, you need to
  - specify the domain,
  - specify the codomain, and
  - give a ***rule*** used to evaluate the function.
- All three pieces are necessary.
  - We need to domain to know what the function can be applied to.
  - We need to codomain to know what the output space is.
  - We need the rule to be able to evaluate the function.
- There are many ways to do this. Let’s go over a few examples.



Functions can be defined as a ***picture***.
Draw the domain and codomain explicitly.
Then, add arrows to show the outputs.

$$f : \mathbb{Z} \to \mathbb{Z}, \text{ where}$$

$$f(x) = x^2 + 3x - 15$$

Functions can be defined as a ***rule***.
Be sure to explicitly state what the domain and codomain are!

$f : \mathbb{Z} \to \mathbb{N}$, where

$$f(n) = \begin{cases} n & \text{if } n \geq 0 \\ -n & \text{if } n \leq 0 \end{cases}$$

Some rules are given ***piecewise***. We select which rule to apply based on the conditions on the right. (Just make sure at least one condition applies and that all applicable conditions give the same result!)

Some Nuances

$$f : \mathbb{R} \to \mathbb{R}, \text{ where}$$

$$f(x) = \frac{x+2}{x+1}$$

Answer at

https://cs103.stanford.edu/pollev

Is this a function from $\mathbb{R}$ to $\mathbb{R}$?

$f : \mathbb{R} \to \mathbb{R}$, where

$$f(x) = \frac{x+2}{x+1}$$

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https://cs103.stanford.edu/pollev

This expression isn't defined when x = -1, so f isn't defined over its full domain. We therefore don't consider it to be a function.

Is this a function from $\mathbb{R}$ to $\mathbb{R}$?

$$f : \mathbb{N} \to \mathbb{R}, \text{ where}$$

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Yep, it's a function! Every natural number maps to some real number.

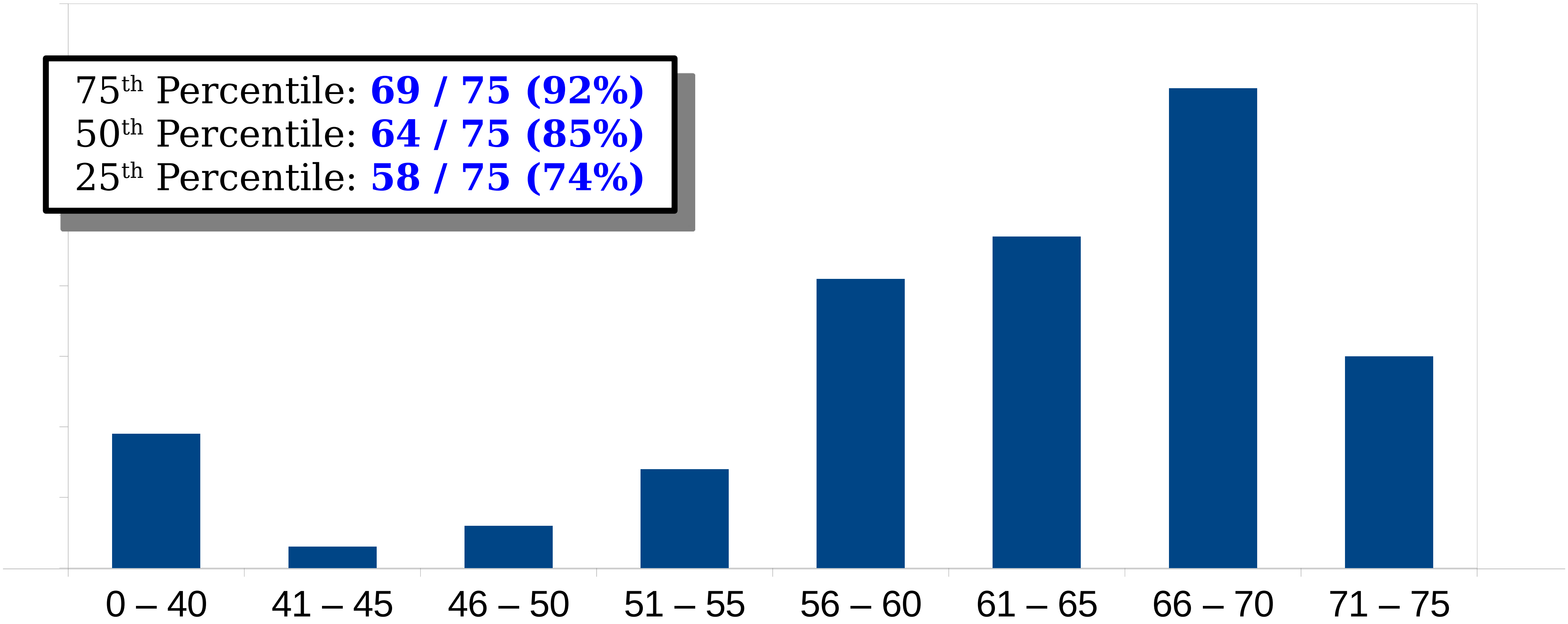
Is this a function from $\mathbb{N}$ to $\mathbb{R}$?

Time-Out for Announcements!

Problem Set One Graded

- Solutions posted on course website (via PS1 page). Check them out!
  - You'll get to see examples of polished written proofs.
  - Each problem has a "Why We Asked This Question" section, which gives some context.
  - We may have solved the problem differently than you, and this will give you more perspectives to use.
- TAs have finished grading Problem Set One!
- Grades and feedback are up on the Gradescope.

Problem Set One Graded

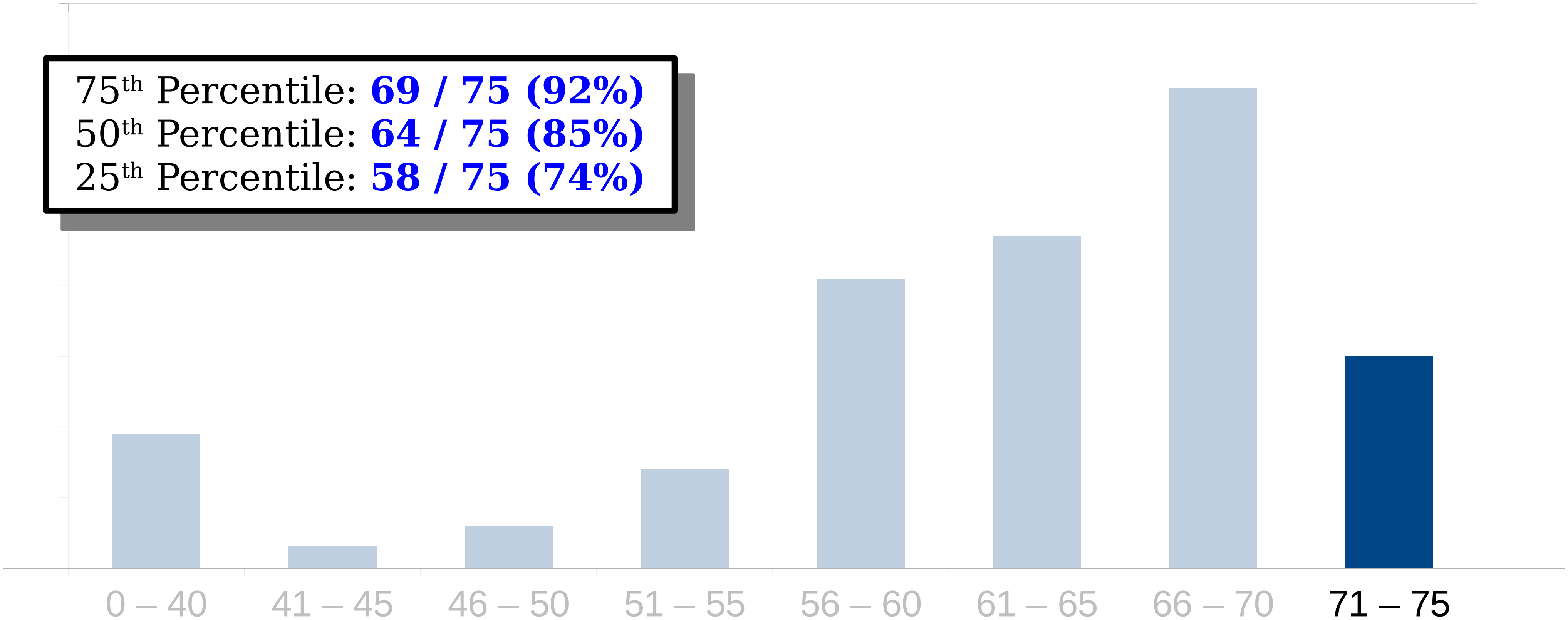


75th Percentile: **69 / 75 (92%)**
50th Percentile: **64 / 75 (85%)**
25th Percentile: **58 / 75 (74%)**

Pro tips when reading a grading distribution:

1. Standard deviations are ***unhelpful and discouraging***. Ignore them.
2. The average score is a ***unhelpful***. Ignore it.
3. Raw scores are ***unhelpful and discouraging***. Ignore them.

Problem Set One Graded



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50th Percentile: **64 / 75 (85%)**
25th Percentile: **58 / 75 (74%)**

"Great job! Look over your feedback for some tips on how to tweak things for next time."

Problem Set One Graded

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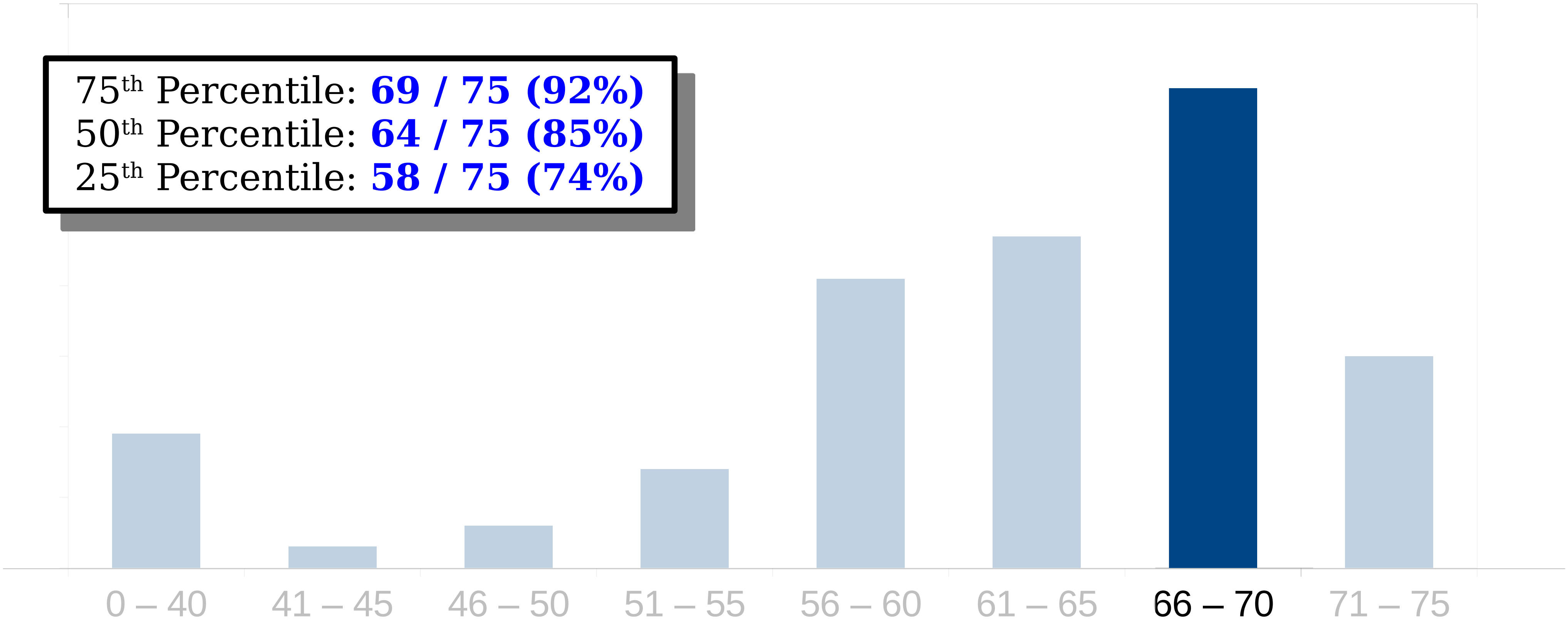
50th Percentile: **64 / 75 (85%)**
25th Percentile: **58 / 75 (74%)**



"You're almost there! Review the feedback on your submission and see what to focus on for next time."

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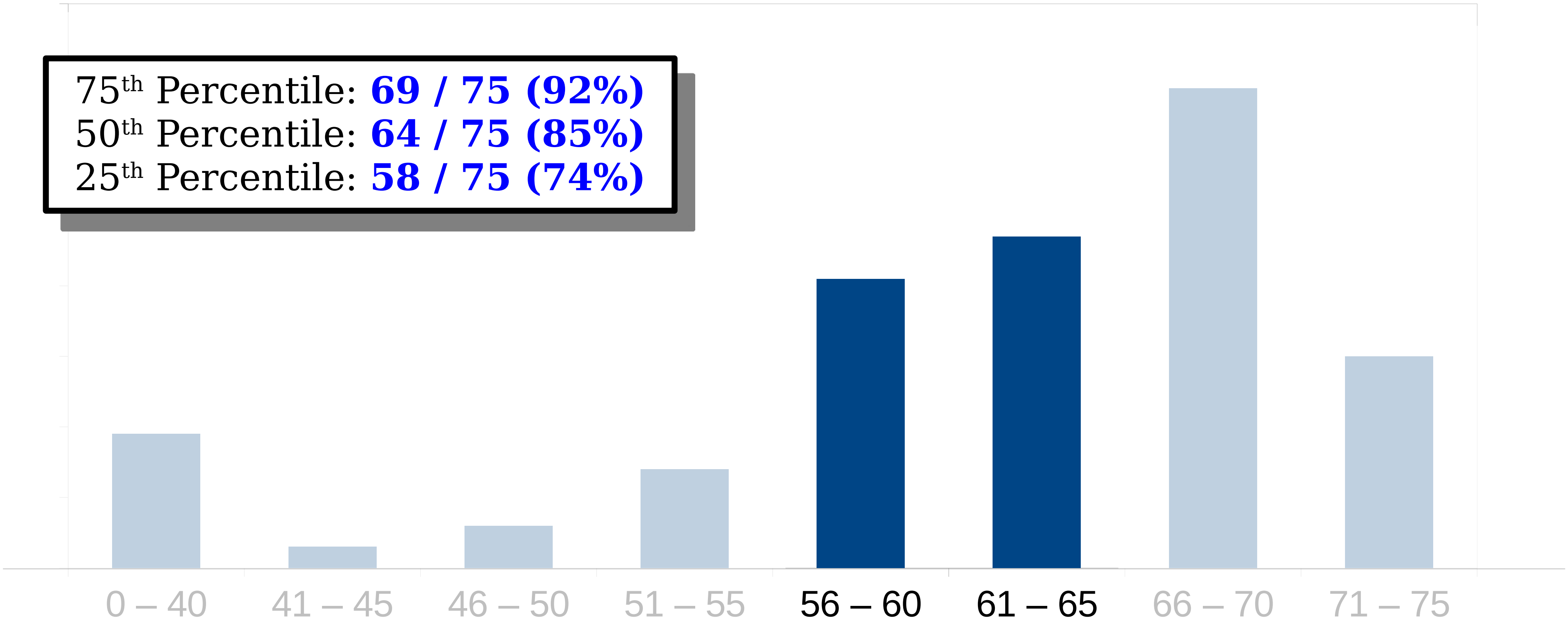

"You're on the right track, but there are some areas where you need to improve. Review your feedback and ask us questions when you have them."

Problem Set One Graded

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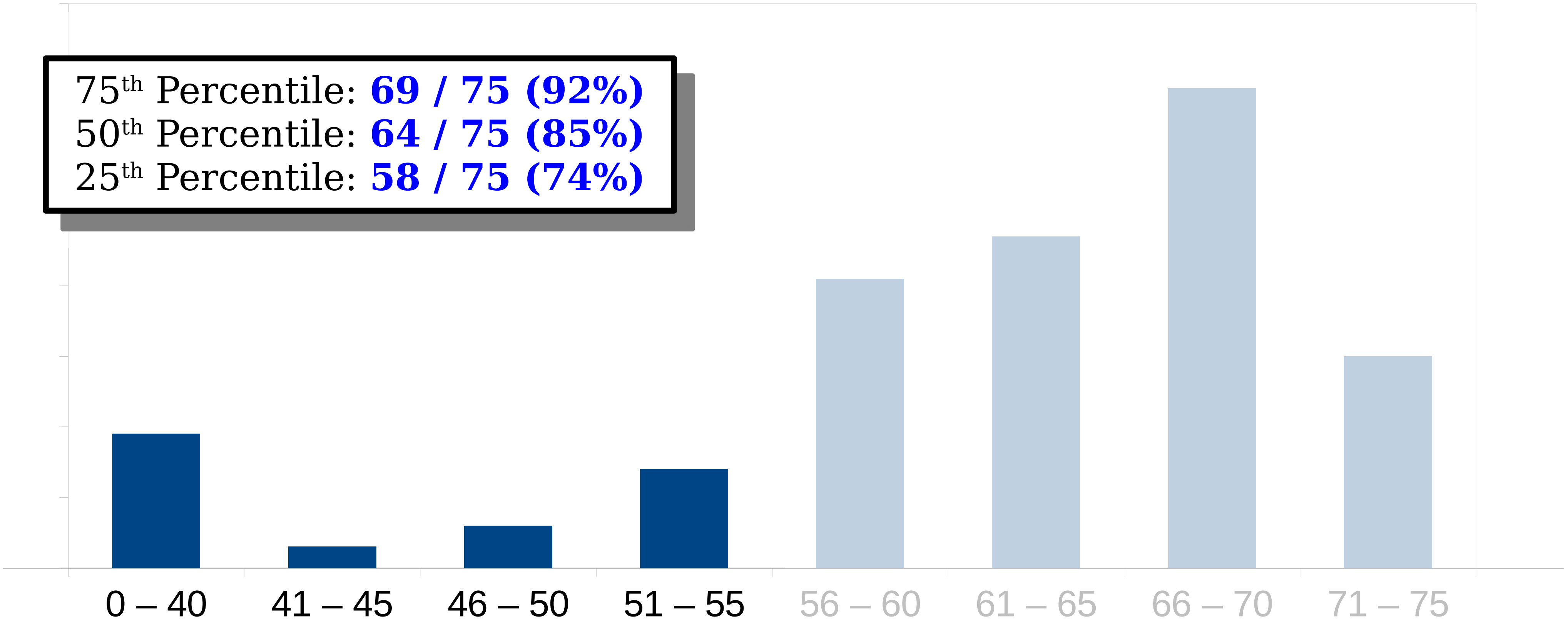


"Looks like something hasn't quite clicked yet. Get in touch with us and stop by office hours to get some extra feedback and advice. Don't get discouraged – you can do this!"

What Not to Think

- “Well, I guess I’m just not good at math.”
  - For most of you, this is your first time doing any rigorous proof-based math.
  - Don’t judge your future performance based on a single data point.
  - Life advice: have a growth mindset!
- “Hey, I did above the median. That’s good enough.”
  - There’s always some area where you can improve. Take the time to see what that is.

Regrade Requests

- We’re human. We make mistakes. And we’re happy to correct them!
- Regrades will open on Gradescope 48 hours after grades are released. They close one week after grades are released.
- Notes on regrades:
  - Please be civil. We make mistakes. We’re happy to correct them.
  - We have to grade what you submitted; we can’t take any clarifications into account during regrades.
  - Regrades are for where we made deductions we shouldn’t have, rather than for the magnitude of deductions.

Essential Action Items

- ***Review your feedback.***
  - Don’t just look at the raw score. Make sure you really, truly understand where you need to improve.
- ***Read the solutions in depth.***
  - Make sure you understand what we were asking, why we asked it, and what we wanted you to take away.
  - (Especially for Q8, Q10) Look at our solutions and see if there’s any neat lessons you can draw from them.
- ***Come to us with questions.***
  - Anything you’re not sure about? That’s what we’re here for! Come to office hours, ask questions on EdStem, etc.

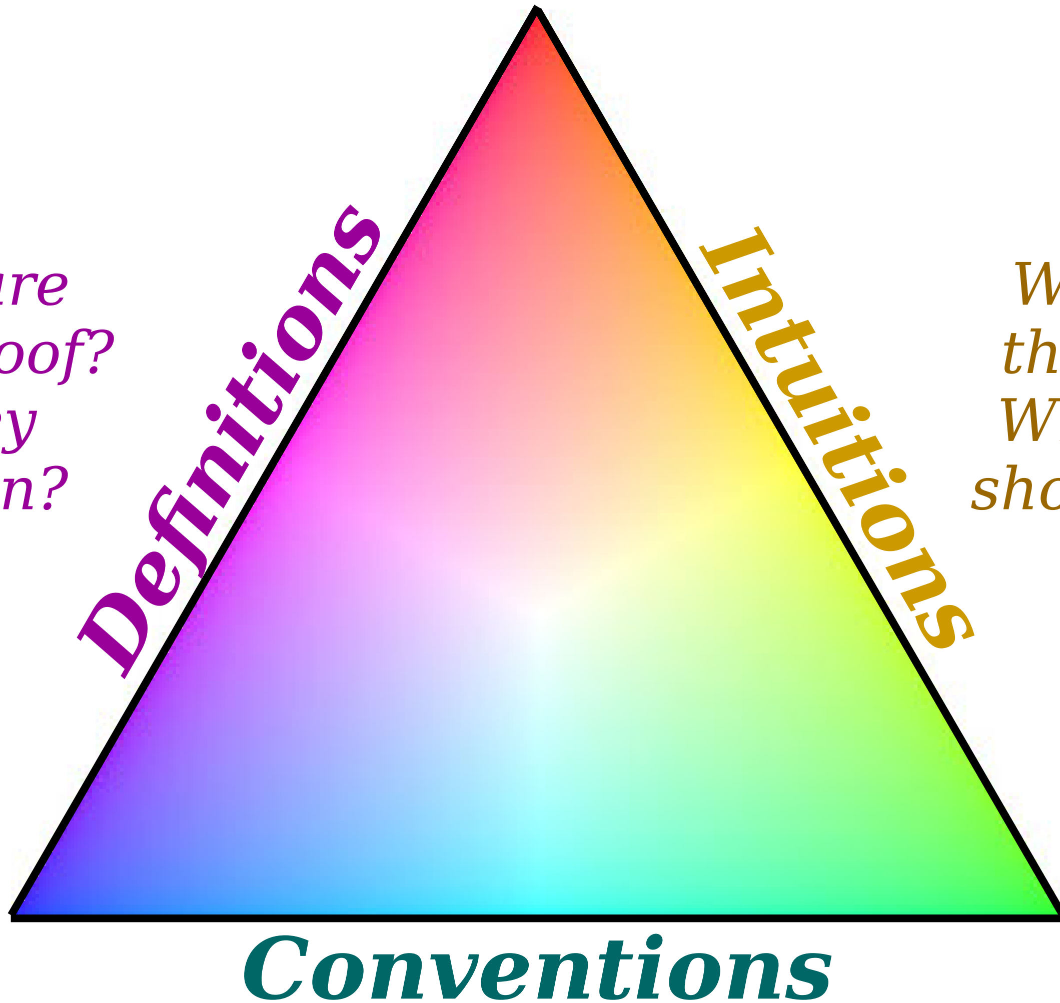
Other Things to Have On Your Radar

- Left-handed desks for exams – form due Monday.
- Attendance opt-out form available next week.
- Bonus opportunity (unlocks Friday or Saturday).
- Regret Clause deadlines (Tuesdays).

Back to CS103!

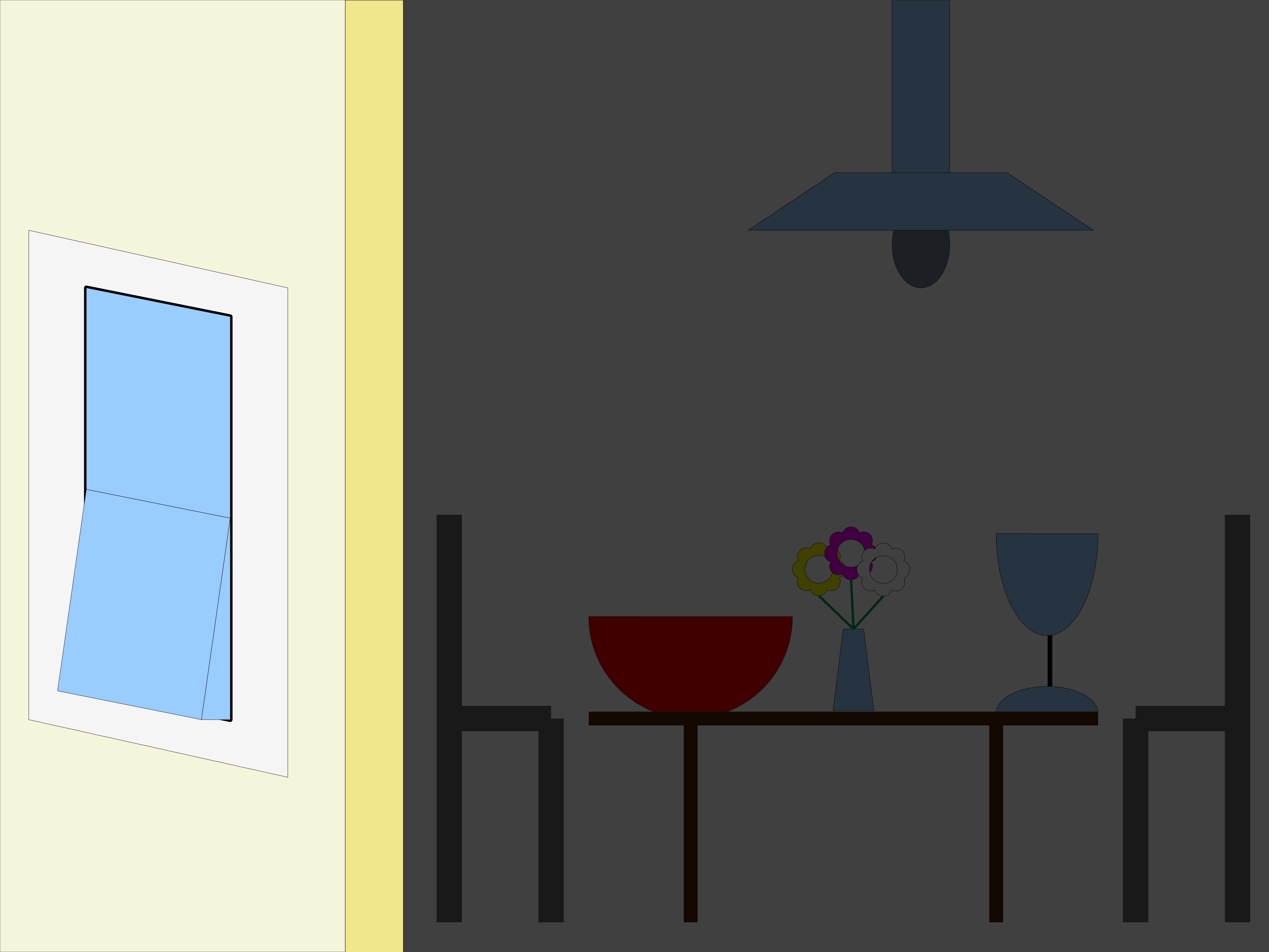
Special Types of Functions

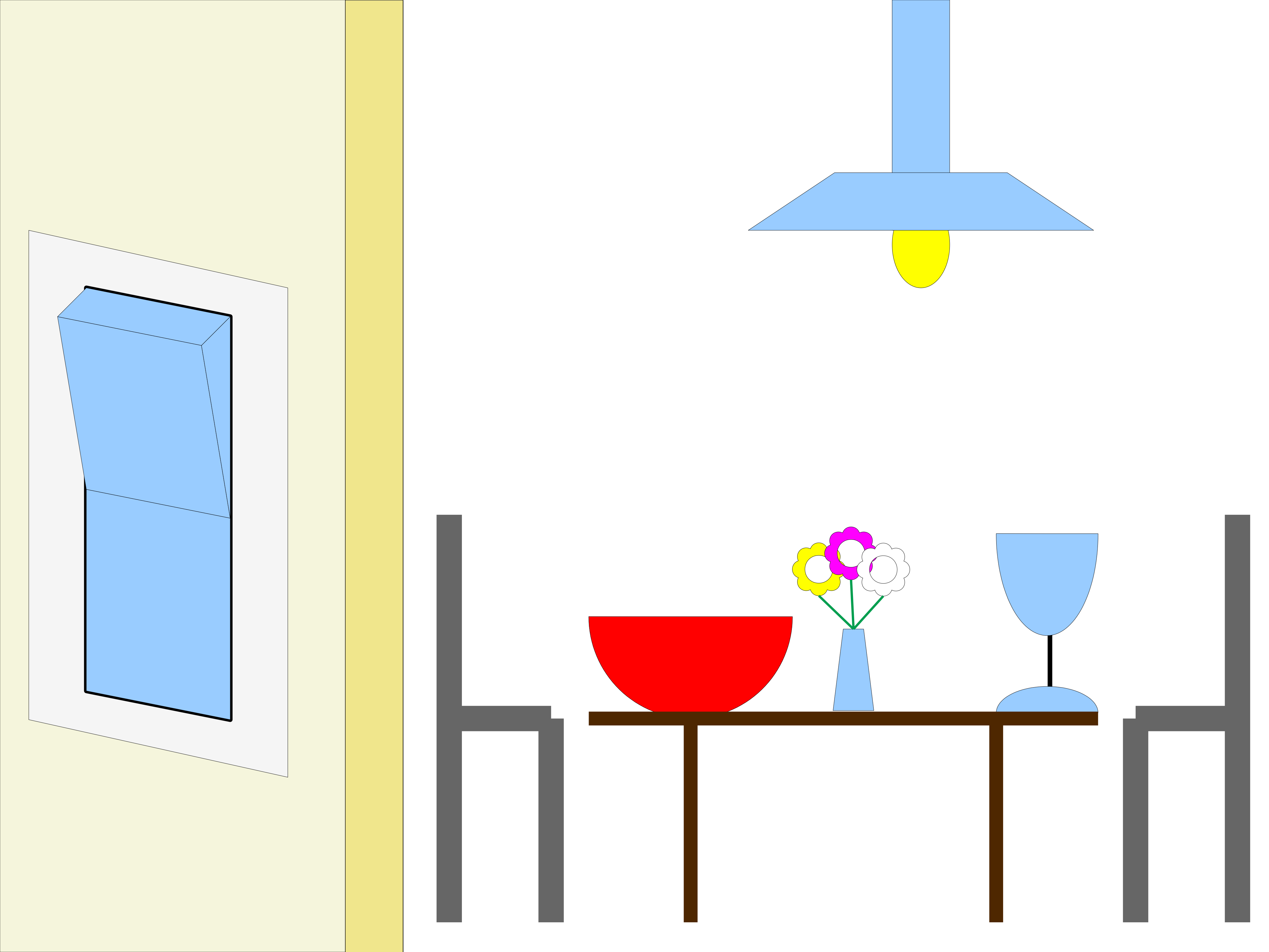
What terms are used in this proof? What do they formally mean?



What does this theorem mean? Why, intuitively, should it be true?

What is the standard format for writing a proof? What are the techniques for doing so?





Undoing by Doing Again

- Some operations invert themselves. For example:
  - Flipping a switch twice is the same as not flipping it at all.
  - In first-order logic, $\neg\neg A$ is equivalent to A.
  - In algebra, $-(-x) = x$.
  - In set theory, $(A \Delta B) \Delta B = A$. *(Yes, really!)*
- Operations with these properties are surprisingly useful in CS theory and come up in a bunch of contexts.
  - Storing compressed approximations of sets (XOR filters).
  - Building encryption systems (symmetric block ciphers).
  - Transmitting a large file to multiple receivers (fountain codes).

Involutions

- A function $f : A \to A$ from a set back to itself is called an ***involution*** when the following first-order logic statement is true about f:

$$\forall x \in A.\ f(f(x)) = x.$$

("Applying f twice is equivalent to not applying f at all.")

- Involutions have lots of interesting properties. Let's explore them and see what we can find.

This is the formal definition. Use it in proofs.

This is just an intuition. Don't use it in proofs.

Involutions

- Which of the following are involutions?
  - $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = x$.
  - $g : \mathbb{Z} \to \mathbb{Z}$ defined as $g(x) = -x$.
  - $h : \mathbb{R} \to \mathbb{R}$ defined as $h(x) = {}^1/_x$.
  - $p : \mathbb{N} \to \mathbb{N}$ defined as follows:

$$p(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

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  - $h : \mathbb{R} \to \mathbb{R}$ defined as $h(x) = 1/x$.
  - $p : \mathbb{N} \to \mathbb{N}$ defined as follows:

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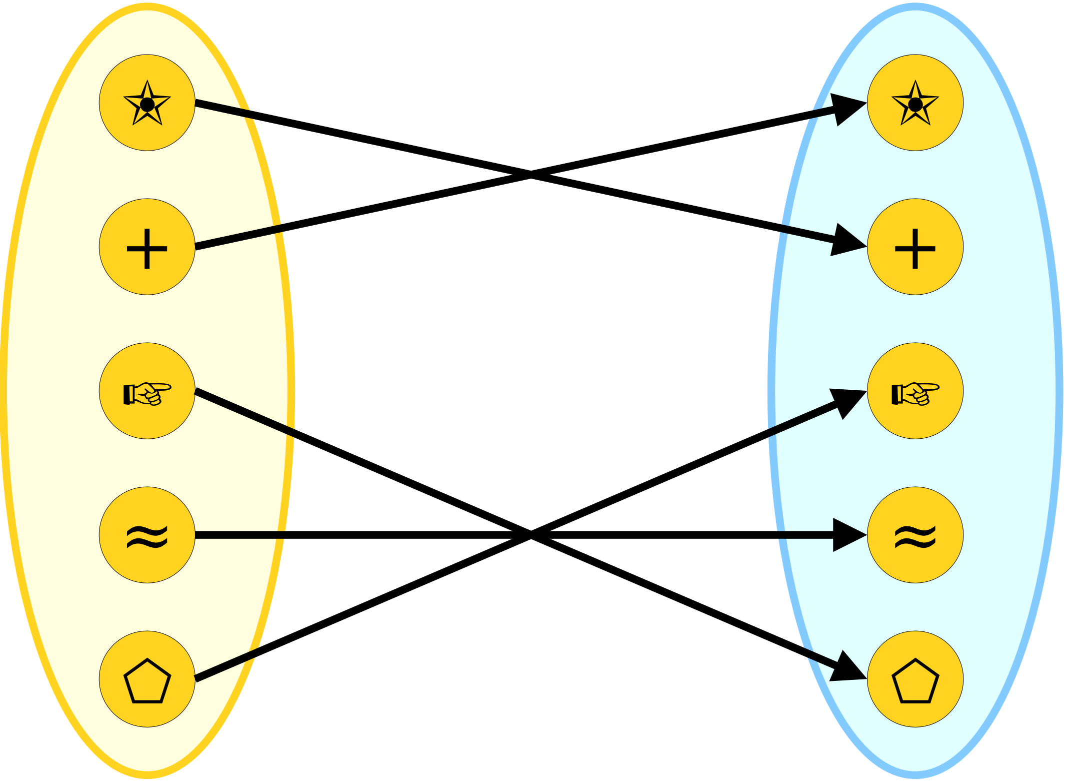
Involutions, Visually



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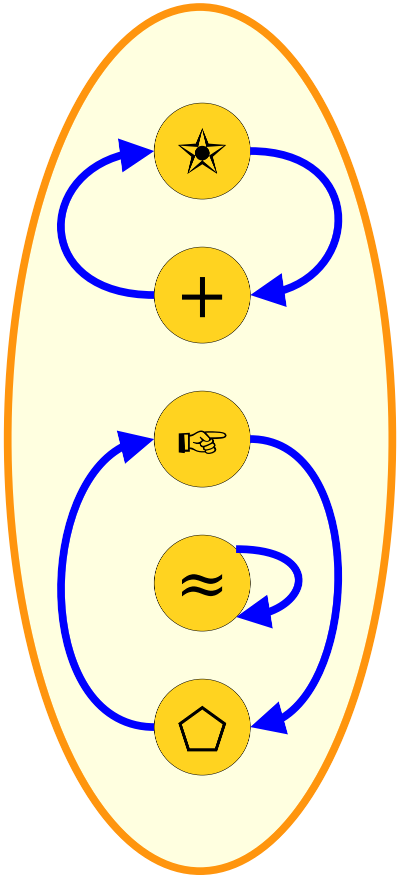
Involutions, Visually



A function $f : A \to A$ is called an ***involution*** if the following first-order logic statement is true about f:

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Proofs on Involutions

Theorem: The function $f : \mathbb{Z} \to \mathbb{Z}$ defined as

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

is an involution.

Theorem: The function $f : \mathbb{Z} \to \mathbb{Z}$ defined as

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This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.

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What is the negation of this statement?

$$\neg \forall n \in \mathbb{N}.\ f(f(n)) = n$$
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Therefore, we need to find some concrete choice of n such that $f(f(n)) \neq n$.

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Pick $n = 2$. Then

$$\begin{aligned} f(f(n)) &= f(f(2)) \\ &= f(4) \\ &= 16, \end{aligned}$$

which means that $f(f(n)) \neq n$, as required. ■

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$\exists x.\ A$		Find an x where A is true. Then prove that A is true for that specific choice of x.
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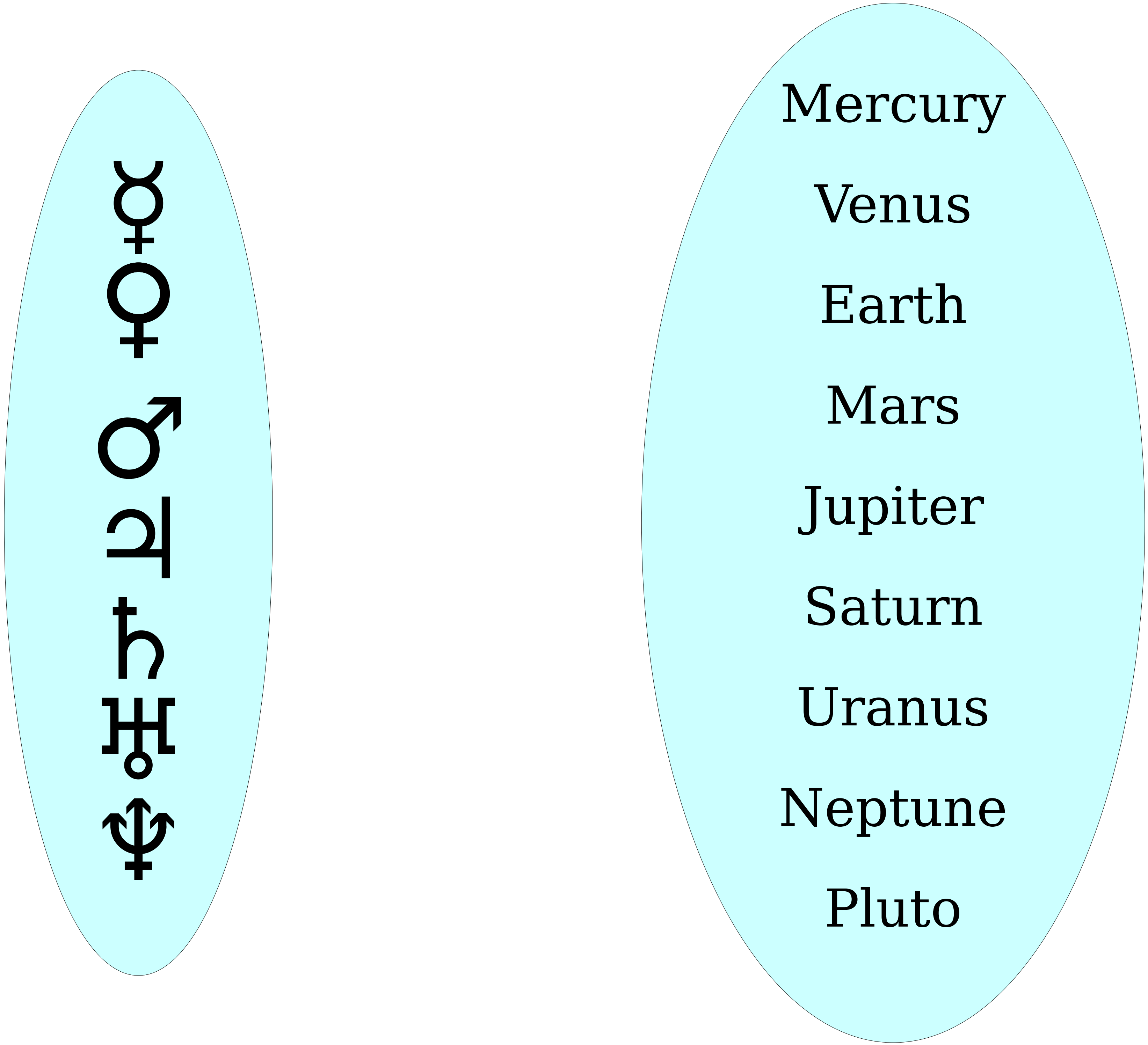
Another Class of Functions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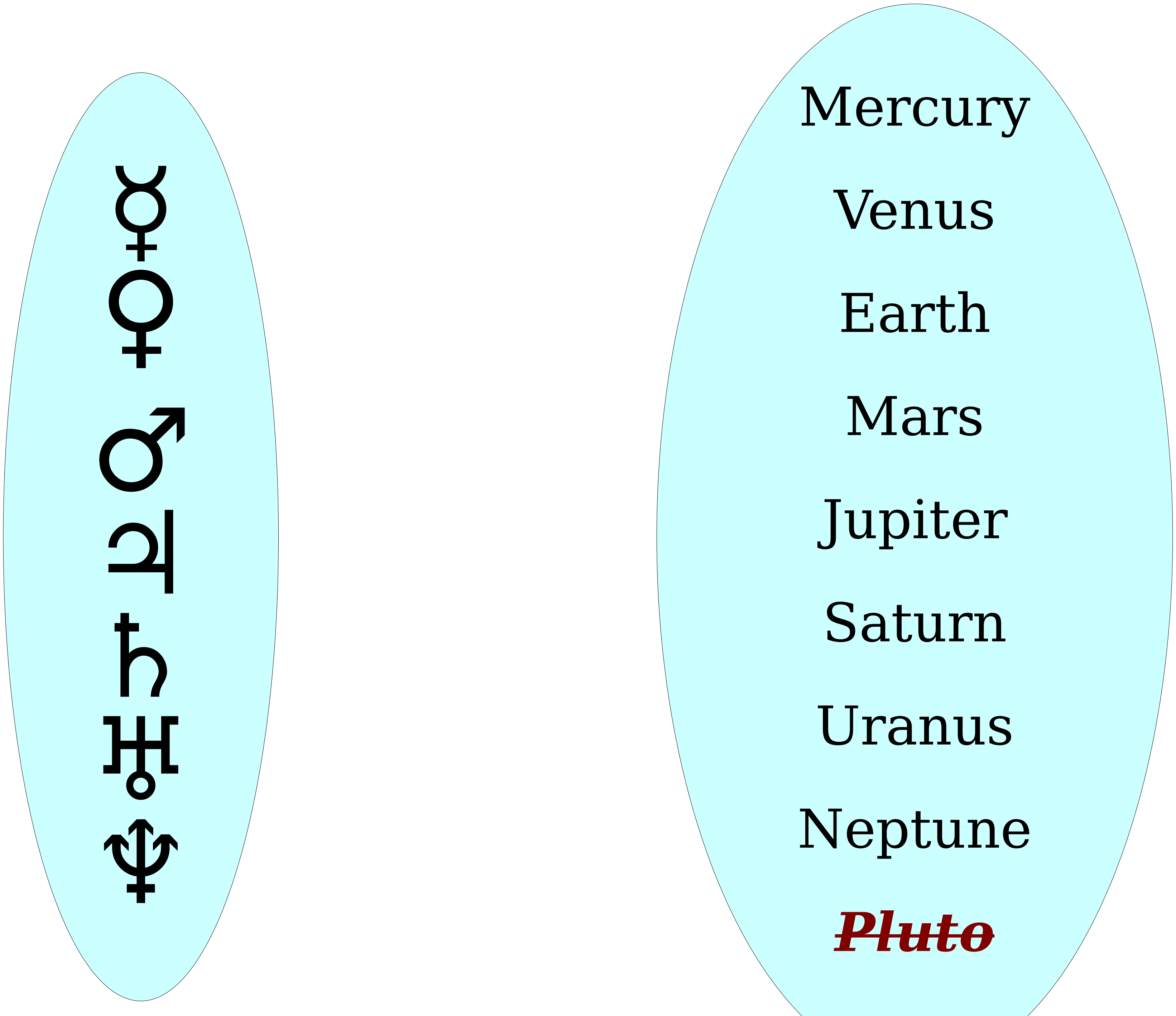
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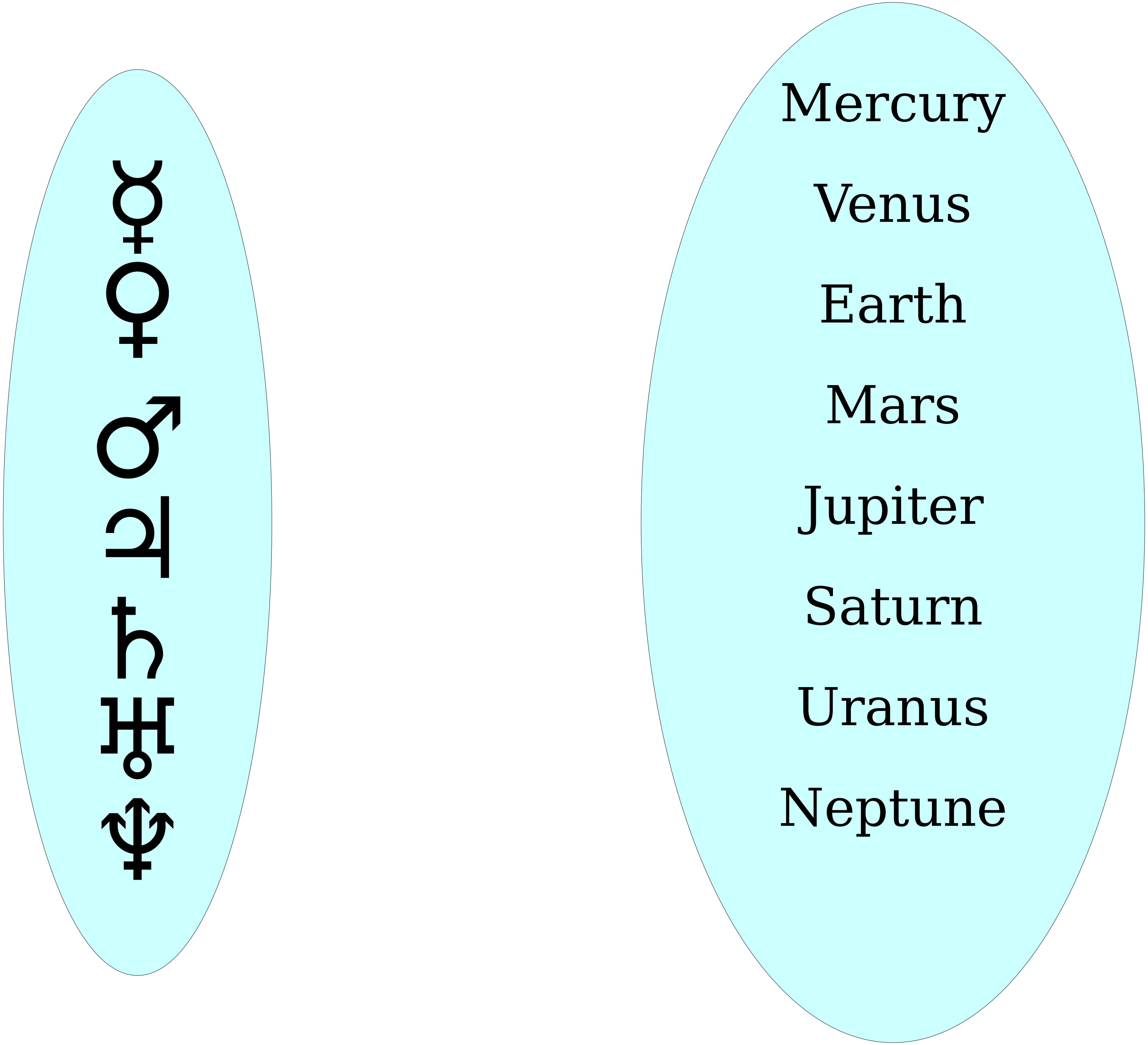
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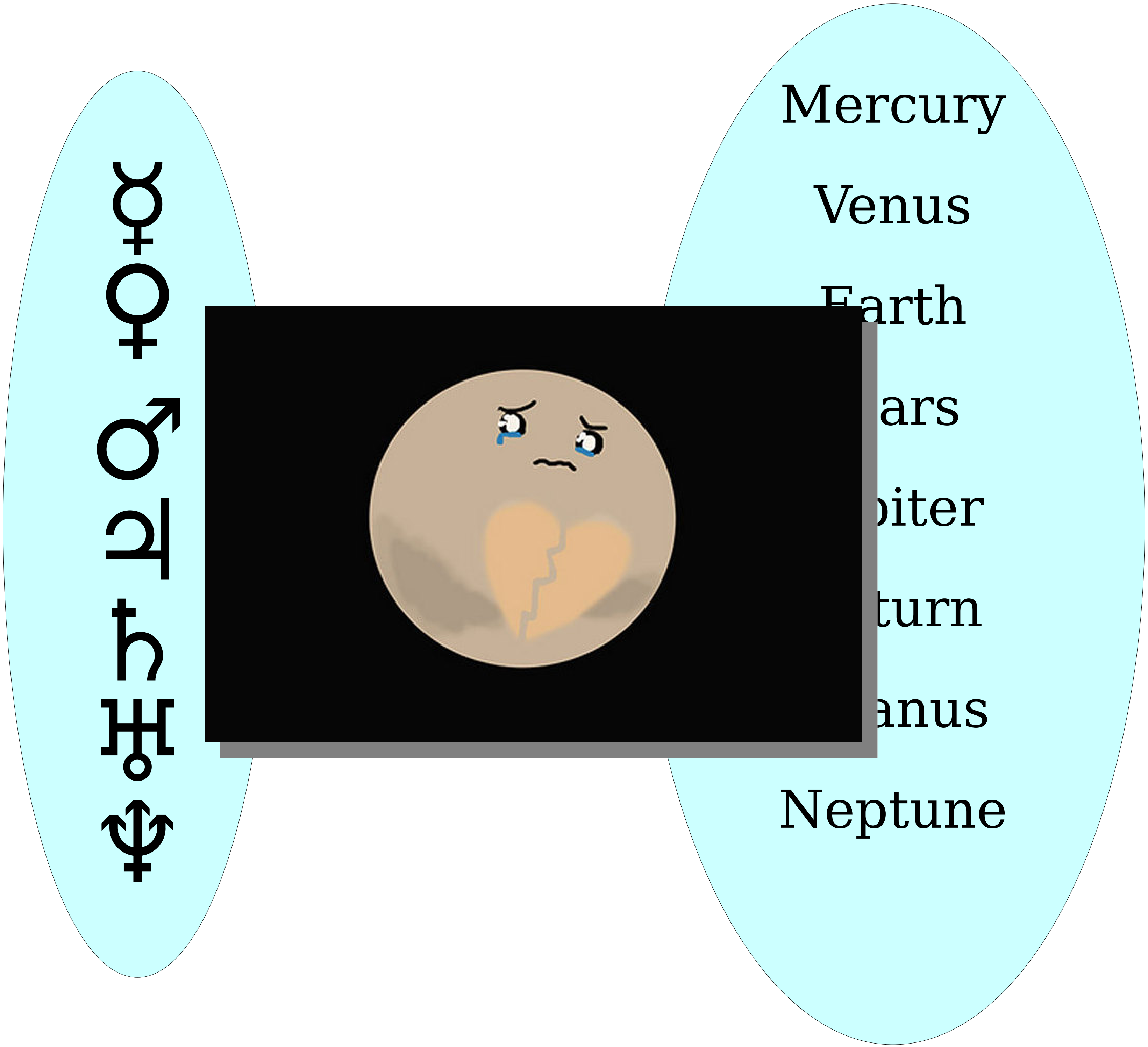
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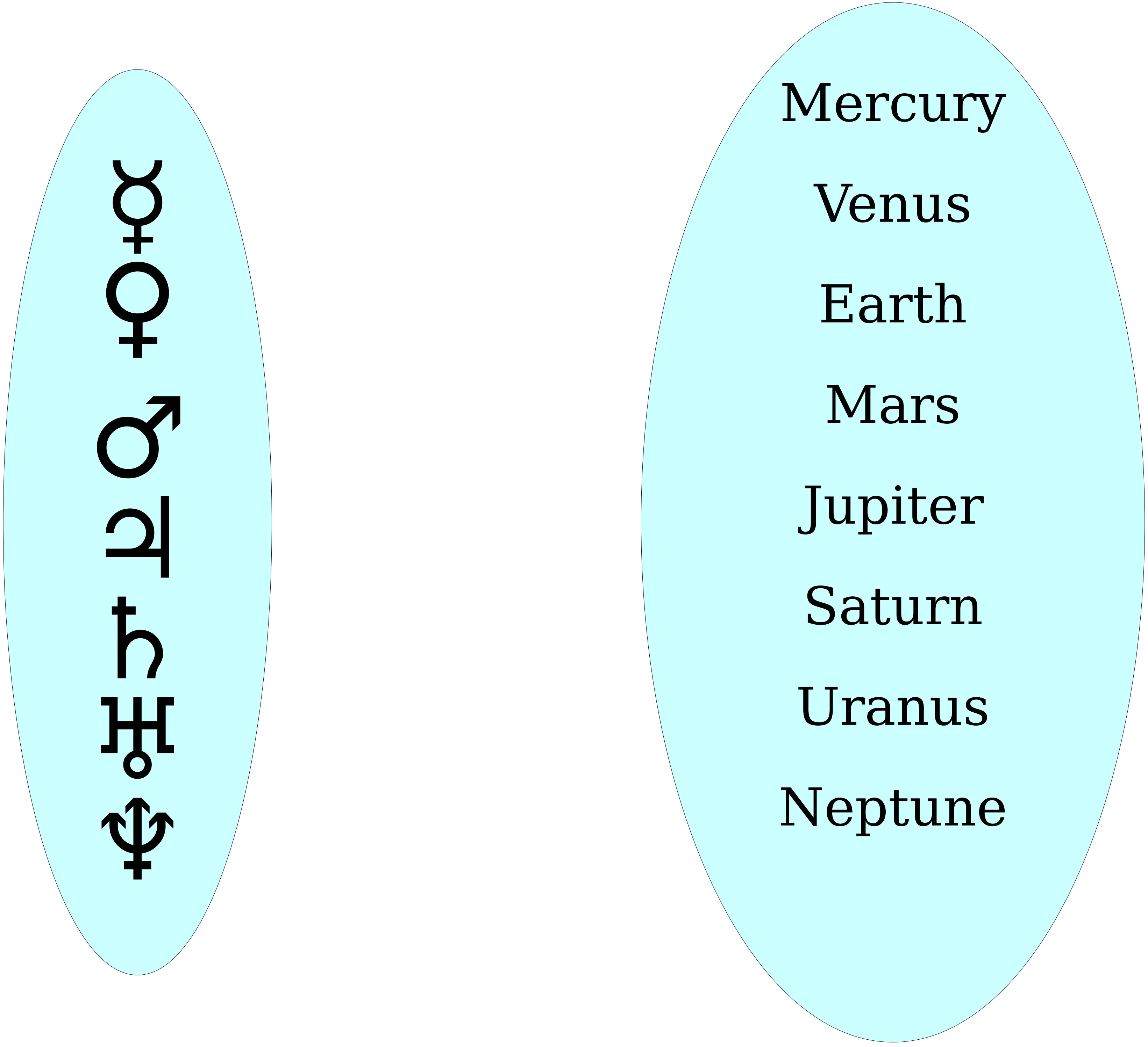
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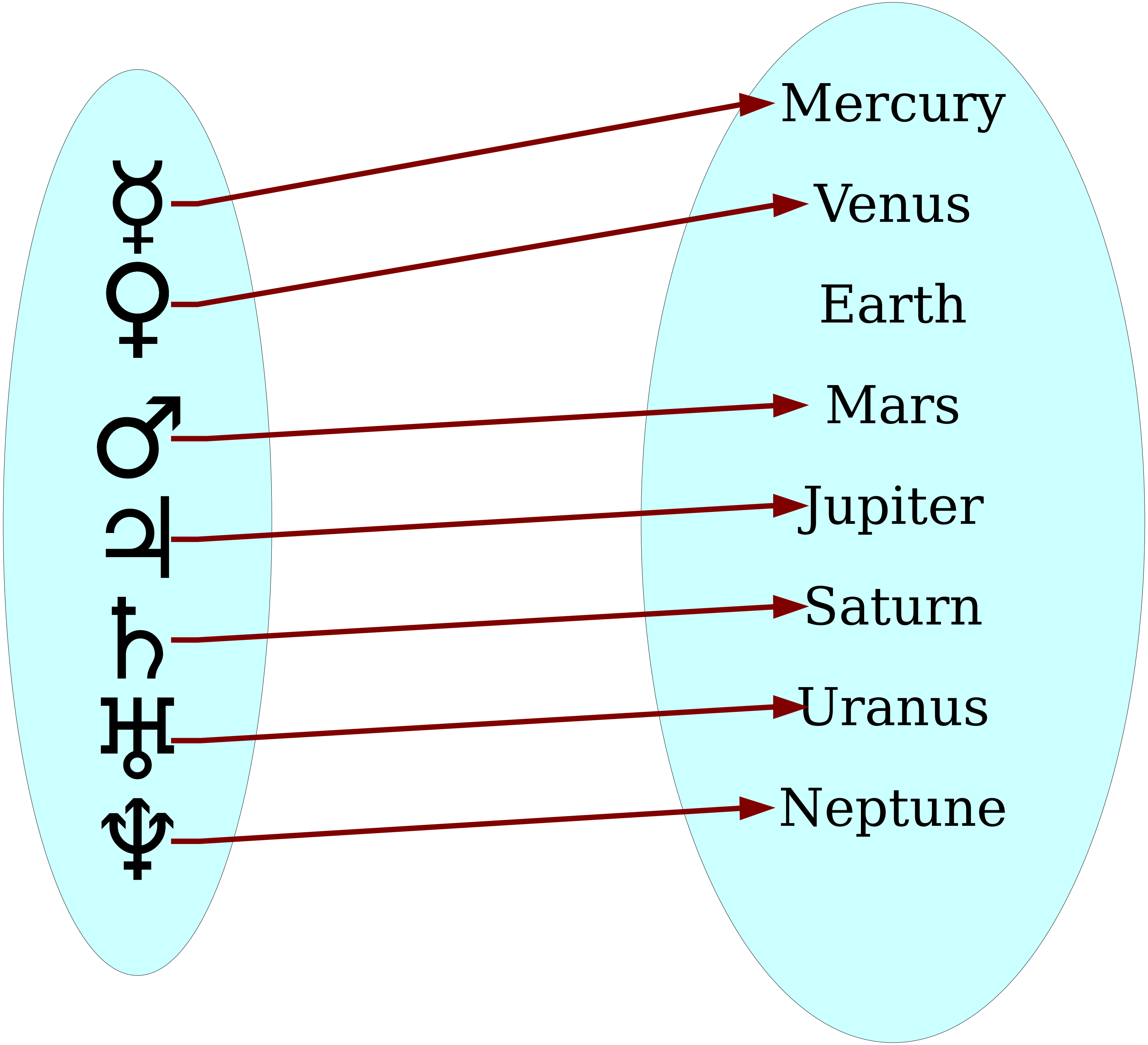
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Injective Functions

- A function $f : A \to B$ is called ***injective*** (or ***one-to-one***) when the following statement is true about f:

$$\forall a_1 \in A. \forall a_2 \in A. (a_1 \neq a_2 \to f(a_1) \neq f(a_2))$$

(*"If the inputs are different, the outputs are different."*)

- The following first-order definition is equivalent *(why?)* and is often useful in proofs.

$$\forall a_1 \in A. \forall a_2 \in A. (f(a_1) = f(a_2) \to a_1 = a_2)$$

(*"If the outputs are the same, the inputs are the same."*)

- A function with this property is called an ***injection***.
- How does this compare to our second rule for functions?

Injections

- Let be the set of all CS103 students. Which of the following are injective?
  - f : → ℕ where $f(x)$ is x's Stanford ID number.
  - g : → , where is the set of all continents and $g(x)$ is x's continent of birth.
  - h : → , where is the set of all given (first) names, where $h(x)$ is x's given (first) name.

Answer at

https://cs103.stanford.edu/pollev

$f : A \to B$ is ***injective*** when either equivalent statement is true:

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Proofs on Injections

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This in turn means that

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Injective Functions

Theorem: Let $f : \mathbb{N} \to \mathbb{N}$ be defined as $f(n) = 2n + 7$. Then f is injective.

Proof: Consider any $n_1, n_2 \in \mathbb{N}$ where $f(n_1) = f(n_2)$. We will prove that $n_1 = n_2$.

Since $f(n_1) = f(n_2)$, we see that

$$2n_1 + 7 = 2n_2 + 7.$$

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Good exercise: Repeat this proof using the other definition of injectivity!

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What does it mean for f to be injective?

$$\forall x_1 \in \mathbb{Z}.\ \forall x_2 \in \mathbb{Z}.\ (x_1 \neq x_2 \to f(x_1) \neq f(x_2))$$

What is the negation of this statement?

$$\neg\forall x_1 \in \mathbb{Z}.\ \forall x_2 \in \mathbb{Z}.\ (x_1 \neq x_2 \to f(x_1) \neq f(x_2))$$
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Theorem: Let $f : \mathbb{Z} \to \mathbb{N}$ be defined as $f(x) = x^4$. Then f is not injective.

Proof: We will prove that there exist integers x_1 and x_2 such that $x_1 \neq x_2$, but $f(x_1) = f(x_2)$.

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Let $x_1 = -1$ and $x_2 = +1$.

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so $f(x_1) = f(x_2)$ even though $x_1 \neq x_2$, as required.

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$A \land B$		Prove A. Also prove B.
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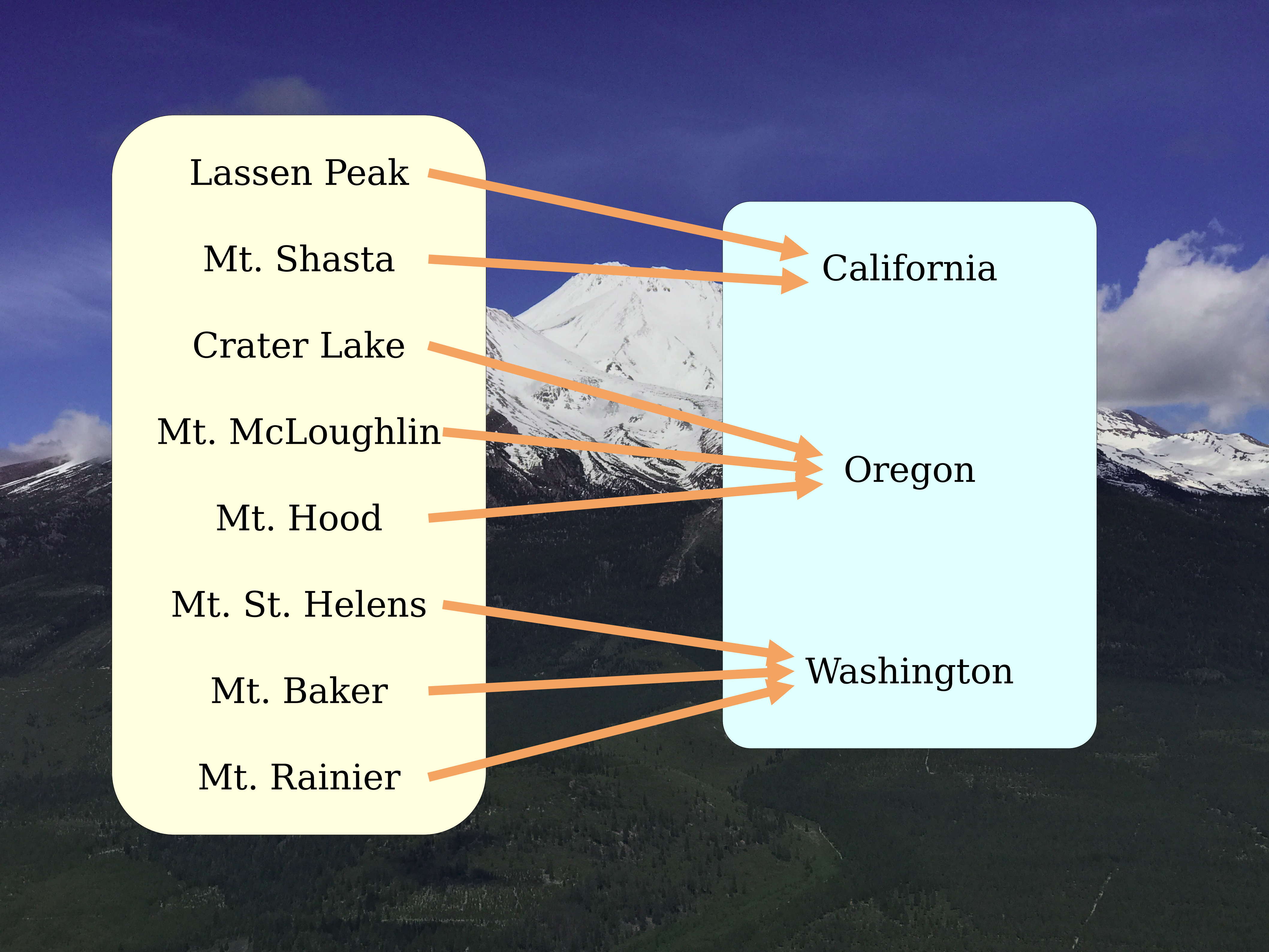
$A \to B$		Assume A is true, then prove B is true.
$A \land B$		Prove A. Also prove B.
$A \lor B$		Either prove $\neg A \to B$ or prove $\neg B \to A$. *(Why does this work?)*
$A \leftrightarrow B$		Prove $A \to B$ and $B \to A$.
$\neg A$		Simplify the negation, then consult this table on the result.

Two More Classes of Functions


Lassen Peak
Mt. Shasta
Crater Lake
Mt. McLoughlin
Mt. Hood
Mt. St. Helens
Mt. Baker
Mt. Rainier
California
Oregon
Washington

Surjective Functions

- A function $f : A \to B$ is called ***surjective*** (or ***onto***) when this first-order logic statement is true about f:

$$\forall b \in B.\ \exists a \in A.\ f(a) = b$$

(*“For every possible output, there's an input that produces it.”*)

- A function with this property is called a ***surjection***.
- How does this compare to our first rule of functions?

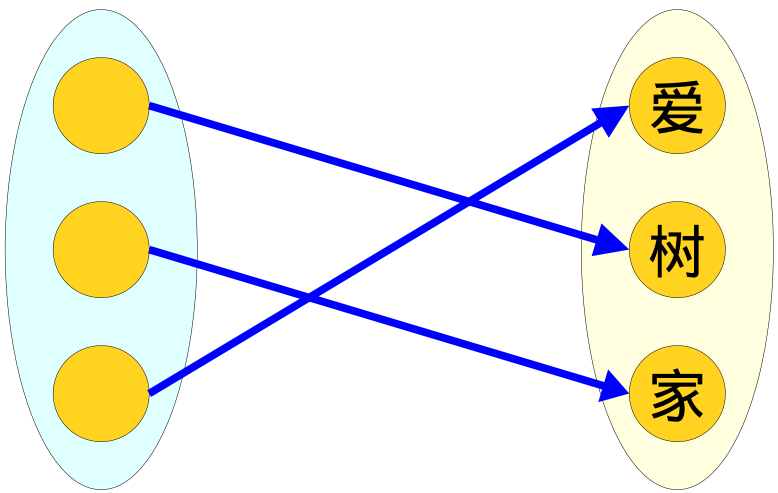
Check the appendix for sample proofs involving surjections.

Injections and Surjections

- An injective function associates ***at most*** one element of the domain with each element of the codomain.
- A surjective function associates ***at least*** one element of the domain with each element of the codomain.
- What about functions that associate ***exactly one*** element of the domain with each element of the codomain?

Bijections

- A ***bijection*** is a function that is both injective and surjective.
- Intuitively, if $f : A \to B$ is a bijection, then f represents a way of pairing off elements of A and elements of B.



Bijections

- Which of the following are bijections?
  - $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = x$.
  - $f : \mathbb{Z} \to \mathbb{R}$ defined as $f(x) = x$.
  - $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = 2x + 1$.
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Next Time

- ***First-Order Assumptions***
  - The difference between assuming something is true and proving something is true.
- ***Connecting Function Types***
  - Involutions, injections, and surjections are related to one another. How?
- ***Function Composition***
  - Sequencing functions together.

Appendix: More Proofs on Functions

Proof 1: Proving a function is surjective.

Surjective Functions

Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 2x$. Then $f(x)$ is surjective.

Surjective Functions

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Proof:

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Proof:

What does it mean for f to be surjective?

$$\forall y \in \mathbb{R}.\ \exists x \in \mathbb{R}.\ f(x) = y$$

Therefore, we'll choose an arbitrary $y \in \mathbb{R}$, then prove that there is some $x \in \mathbb{R}$ where $f(x) = y$.

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Proof:

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Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 2x$. Then $f(x)$ is surjective.

Proof: Consider any $y \in \mathbb{R}$.

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Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 2x$. Then $f(x)$ is surjective.

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Let $x = y / 2$.

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Let $x = y / 2$. Then we see that

$$f(x) = f(y / 2)$$

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So we see that $f(x) = y$, as required. ■

This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.

Proof 2: Proving a function is not surjective.

Surjective Functions

Theorem: Let $g : \mathbb{N} \to \mathbb{N}$ be defined as $g(n) = 2n$. Then $g(x)$ is not surjective.

Surjective Functions

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What does it mean for g to be surjective?

$$\forall n \in \mathbb{N}.\ \exists m \in \mathbb{N}.\ g(m) = n$$

What is the negation of the above statement?

$$\neg\forall n \in \mathbb{N}.\ \exists m \in \mathbb{N}.\ g(m) = n$$
$$\exists n \in \mathbb{N}.\ \neg\exists m \in \mathbb{N}.\ g(m) = n$$
$$\exists n \in \mathbb{N}.\ \forall m \in \mathbb{N}.\ g(m) \neq n$$

Therefore, we need to find a natural number n where, regardless of which m we pick, we have $g(m) \neq n$.

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Our overall goal is to prove

$$\exists n \in \mathbb{N}.\ \forall m \in \mathbb{N}.\ g(m) \neq n.$$

We just made our choice of n. Therefore, we need to prove

$$\forall m \in \mathbb{N}.\ g(m) \neq n.$$

We'll therefore pick an arbitrary $m \in \mathbb{N}$, then prove that $g(m) \neq n$.

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This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.